

# New Bounds on False-Name Manipulation in Weighted Voting Games

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## Abstract

We consider *false-name manipulation* in *weighted voting games* (WVGs). False-name manipulation involves an agent, termed a *manipulator*, splitting its weight among several false identities in anticipation of power increase. False-name manipulation has been identified as a problem in WVGs. Indeed, in open anonymous environments, this manipulation can be easy and cheap to achieve. Previous works have considered false-name manipulation in WVGs using the well-known *Shapley-Shubik* and *Banzhaf* indices to compute agents' power. Upper and lower bounds on the extent of power that a manipulator may gain exist for the case when a manipulator splits into  $k = 2$  false identities for both the *Shapley-Shubik* and *Banzhaf* indices. The bounds on the case when an agent splits into  $k > 2$  false identities, until now, has remained open for the two indices. This paper answers this open problem by providing *four* non-trivial bounds on false-name manipulation in WVGs when an agent splits into  $k > 2$  false identities for both the *Shapley-Shubik* and *Banzhaf* indices. One of the bounds is also shown to be asymptotically tight, i.e., there exists at least a game in which an agent achieves the proposed bound by splitting into several false identities.

## 1 Introduction

WVGs are classic cooperative games which provide compact representation for coalition formation models in human societies and multiagent systems. One way of modeling cooperation for making joint decisions that is frequently found in the real world is via the use of WVGs. WVGs represent mathematical abstractions of voting systems. In a WVG, a *quota* is given and each agent has an associated *weight*. A subset of agents whose total weight is at least the value of the quota is called a *winning coalition*. Agents' relative power in such games is measured using *power indices*. Two prominent indices found in the literature are the *Shapley-Shubik* (Shapley and Shubik 1954) and *Banzhaf* (Banzhaf 1965) indices. These indices are used in this paper to analyze the effects of false-name manipulation in WVGs.

*False-name manipulation* in WVGs, originally studied by (Bachrach and Elkind 2008), involves an agent, termed a *manipulator*, splitting its weight among several identities (called *false agents*) in anticipation of power increase. False-name manipulation has been identified as a problem in

WVGs (Bachrach and Elkind 2008; Aziz and Paterson 2009; Aziz et al. 2011). This is because the anticipated power increase by a manipulator is at the expense of other agents in the game. When an agent engages in false-name manipulation, the number of agents in the new game increases by the number of false agents that the manipulator splits. Also, the quota and weights of other agents in the new game remain the same. The sum of the power of the false agents becomes the new power of the manipulator.

Bachrach and Elkind (2008) and Aziz and Paterson (2009) study false-name manipulation in WVGs using the *Shapley-Shubik* and *Banzhaf* indices respectively to evaluate the effect of this problem when an agent splits into *exactly*  $k = 2$  false identities. They provide upper and lower bounds on the extent of power a manipulator may gain or lose in a WVG. The two papers, however, left as an open problem bounds on the case when an agent splits into  $k > 2$  false identities. Apart from this, Lasisi and Allan (2010) consider an empirical analysis of splitting into more than two false identities but left the issue of theoretical bounds as an open problem. This problem, until now, also remain open in a recent work of (Aziz et al. 2011). We answer this open problem by providing four non-trivial bounds on false-name manipulation in WVGs when a manipulator splits into  $k > 2$  false identities using the *Shapley-Shubik* and *Banzhaf* power indices to compute agents' power.

## 2 Preliminaries

**Definition 1.** *Simple Game.*

Let  $I = \{1, \dots, n\}$  be a set of  $n \in \mathbb{N}$  agents. The non-empty subsets of  $I$  are called *coalitions*. A simple game is a coalitional game,  $(I, v)$ , where  $v : 2^I \rightarrow \{0, 1\}$ . A coalition  $S \subseteq I$  is *winning* if  $v(S) = 1$  and *losing* if  $v(S) = 0$ .

**Definition 2.** *Weighted Voting Game.*

A WVG is a simple game which has a weighted form,  $(W, q)$ , where  $W = (w_1, \dots, w_n) \in (\mathbb{R}^+)^n$  corresponds to the weights of agents in  $I$ , and  $q \in \mathbb{R}^+$  is the quota of the game. A coalition  $S$  wins if the total weight of  $S$ ,  $w(S) = \sum_{i \in S} w_i \geq q$ , which implies that  $v(S) = 1$ . A WVG  $G$  of  $n$  agents with quota  $q$  is denoted by  $G = [w_1, \dots, w_n; q]$ .

**Definition 3.** *Shapley-Shubik Power Index.*

The *Shapley-Shubik* index quantifies the marginal contribution of an agent to the grand coalition. Each permutation of

the agents is considered. We term an agent *pivotal* in a permutation if the agents preceding it do not form a winning coalition, but by including this agent, a winning coalition is formed. We specify the computation of the index using notation of (Aziz et al. 2011). Denote by  $\pi$ , a permutation of the agents, so  $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ , and by  $\Pi$  the set of all possible permutations. Denote by  $S_\pi(i)$  the predecessors of agent  $i$  in  $\pi$ , i.e.,  $S_\pi(i) = \{j : \pi(j) < \pi(i)\}$ . The Shapley-Shubik index,  $\varphi_i(G)$ , for each agent  $i$  in a WVG  $G$ :

$$\varphi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))]. \quad (1)$$

**Definition 4.** *Banzhaf Power Index.*

An agent  $i \in S \subseteq I$  is referred to as being *critical* in a winning coalition,  $S$ , if  $w(S) \geq q$  and  $w(S \setminus \{i\}) < q$ . The Banzhaf power index computation for an agent  $i$  is the proportion of times  $i$  is critical compared to the total number of times any agent in the game is critical. The Banzhaf index,  $\beta_i(G)$ , for each agent  $i$  in a WVG  $G$  is given by

$$\beta_i(G) = \frac{\eta_i(G)}{\sum_{j \in I} \eta_j(G)} \quad (2)$$

where  $\eta_i(G)$  is the number of coalitions for which agent  $i$  is critical in game  $G$ .

### Formal Problem Definition

Let  $k \in \mathbb{N}$ . Consider a WVG  $G = [w_1, \dots, w_n; q]$  of  $n$  agents. Let agent  $i \in I$  with weight  $w_i$  in  $G$  be a manipulating agent. Suppose agent  $i$  splits its weight among  $k \geq 2$  false agents,  $i_1, \dots, i_k$ , having corresponding weights,  $w_{i_1}, \dots, w_{i_k}$ , such that  $w_i = \sum_{j=1}^k w_{i_j}$  and  $w_{i_j} > 0$ . We have a new set of agents after splitting:  $I' = \{1, \dots, i-1, i_1, \dots, i_k, i+1, \dots, n\}$ . The initial game  $G$  of  $n$  agents has been altered by agent  $i$  to give a new WVG  $G'$  of  $n+k-1$  agents. Note that the weights of other agents and the quotas in the two games are the same.

Let  $\phi$  be any of Shapley-Shubik or Banzhaf index. Denote by  $(\phi_1(G), \dots, \phi_n(G)) \in [0, 1]^n$  the power of agents in a WVG  $G$  of  $n$  agents. Thus, for a manipulating agent  $i$  with power  $\phi_i(G)$  in  $G$ , the sum of the power of the  $k \geq 2$  false agents in  $G'$  that the manipulating agent splits is  $\sum_{j=1}^k \phi_{i_j}(G')$ . The ratio  $\tau = \frac{\sum_{j=1}^k \phi_{i_j}(G')}{\phi_i(G)}$  compares the sum of the power of the false agents in the altered game  $G'$  to the power of the manipulator (before it splits) in the original game  $G$ .  $\tau$  gives a factor of the power gained or lost when agent  $i$  alters game  $G$  to give  $G'$ . We say that  $\phi$  is susceptible to manipulation if there exists a game  $G'$  such that  $\tau > 1$ ; the split is termed *advantageous*. If  $\tau < 1$ , then the split is *disadvantageous*, while the split is *neutral* when  $\tau = 1$ .

### 3 Shapley-Shubik Power Index Bounds

**Theorem 1.** (Upper Bound). *Let  $G$  be a WVG of  $n$  agents. If an agent  $i$  alters  $G$  by splitting into  $k \geq 2$  false agents in a new game  $G'$ , then, the power index of the agent in the new game using the Shapley-Shubik index is at most  $\frac{nk}{n+k-1} \varphi_i(G)$ . Moreover, this bound is asymptotically tight.*

*Proof.* Let an agent  $i$  be a distinguished manipulator that splits into  $k$  false agents,  $i_1, \dots, i_k$ . Let  $\Pi_{G-i}$  be the set of all permutations of the remaining  $n-1$  agents in game  $G$  (i.e., not including agent  $i$ ). Also, let  $r \in \mathbb{N}$  be the positions of the agents in permutations  $\Pi_{G-i}$ . We define the term, *i-pivotal-basis*, to be a permutation  $\pi_r \in \Pi_{G-i}$  such that it is possible to insert agent  $i$  into  $\pi_r$  immediately after position  $r$  to make  $i$  pivotal in game  $G$ . We refer to the resulting permutation(s) as being *i-pivotal*. For example, consider a WVG  $[5, 5, 4, 1; 12]$ , having the agent, say  $i$ , with weight 4 as a manipulator. The agent can be inserted into the *i-pivotal-basis* permutation  $(5, 5, 1)$  in two different ways: before and after 1, and resulting in the *i-pivotal* permutations  $(5, 5, 4, 1)$  and  $(5, 5, 1, 4)$ .

Define also  $\pi'_r$  to be a *morphed* permutation in game  $G'$  as one formed from inserting the false agents,  $i_1, \dots, i_k$ , into permutation  $\pi_r$  starting from the  $r$ -th position in any order. Suppose there are  $P_i$  *i-pivotal* different permutations that can be formed from the set  $\Pi_{G-i}$ , then, the Shapley-Shubik index of agent  $i$  in game  $G$ ,  $\varphi_i(G) = \frac{P_i}{n!}$ .

Our hope is that for every case where the insertion of agent  $i$  into a permutation  $\pi_r$  at position  $r$  makes  $i$  pivotal in game  $G$  we can create  $X \in \mathbb{N}$  permutations in which a false agent is pivotal in the corresponding morphed permutation  $\pi'_r$  of game  $G'$ . There are two ways we can obtain permutations in game  $G'$  in which one of the false agents is pivotal:

- Insert the false agents,  $i_1, \dots, i_k$ , into a permutation in game  $G$  for which agent  $i$  is not pivotal, but a false agent is now pivotal in the altered game  $G'$ .
- Insert the false agents,  $i_1, \dots, i_k$ , into an *i-pivotal-basis* permutation  $\pi_r$  starting immediately after the  $r$ -th position. Those seem like good candidates, because the resulting permutation after inserting  $i$  into  $\pi_r$  immediately after position  $r$  is *i-pivotal*, so a false agent may be pivotal in the morphed permutation  $\pi'_r$  of game  $G'$ .

**Case A:** We show that there are no permutations in this case. Suppose there exists a morphed permutation of game  $G'$  in which a false agent  $i_j$  is pivotal, but the permutation  $\pi_r^* \in \Pi_{G-i}$  in game  $G$  from which it is formed is not *i-pivotal-basis*. This permutation in  $G'$  has the following form:

$$C \quad i_j \quad D$$

where,  $C$  and  $D$ , are respectively the left and right sides of the morphed permutation from the pivotal false agent  $i_j$ . Taking this permutation, create a new permutation by sliding all the false agents from  $C$  right towards  $i_j$ . Also, slide all the false agents from  $D$  left towards  $i_j$ . Now, all the false agents occur together with  $i_j$  still pivotal in this new permutation. This shows that the insertion of agent  $i$  into the permutation  $\pi_r^*$  makes  $i$  pivotal in game  $G$ . Thus,  $\pi_r^*$  is *i-pivotal-basis*. Since this is a contradiction to our assumption, there are no permutations in this case.

**Case B:** Consider a certain *i-pivotal-basis* permutation  $\pi_r \in \Pi_{G-i}$  for which the insertion of agent  $i$  immediately after position  $r$  makes  $i$  pivotal in  $G$ . We need to insert the false agents into this permutation starting immediately after position  $r$ . This is done by the following steps:

1. Decide on which of the  $k$  false agents should be pivotal in the newly created permutation. There are  $C(k, 1) = k$  ways<sup>1</sup> of doing this.
2. Order the remaining false agents, and call this ordering,  $\rho$ . There are  $(k-1)!$  ways of doing this.
3. Now, merge  $\rho$  with  $\pi_r$  without changing the order of elements in  $\rho$  or  $\pi_r$  to create a new permutation  $\pi'_r$ . To understand how to count the ways of doing this, realize that there are  $n-1$  items in  $\pi_r$  and  $k-1$  items in  $\rho$ . To get a complete ordering, we form a new permutation by taking the next element of  $\pi_r$  or  $\rho$ . This is simply permutations with repetition<sup>2</sup>, which gives,  $\frac{(n-1+k-1)!}{(n-1)!(k-1)!}$ .
4. To complete the new permutation  $\pi'_r$ , we place the agent selected in step 1 at the appropriate spot in the permutation. There is at most one possibility. Either we find a place to insert the agent to make it pivotal, or we cannot.

Now, from steps 1 to 4 above, we see that there are at most  $\frac{k \cdot (k-1)! \cdot (n-1+k-1)!}{(n-1)!(k-1)!} = \frac{k \cdot (n+k-2)!}{(n-1)!}$  possible ways of finding a new permutation in which a false agent is pivotal in  $\pi'_r$ . We repeat the process for each of the  $P_i$   $i$ -pivotal-basis permutations  $\pi_r \in \Pi_{G-i}$ . Hence, the sum of the Shapley-Shubik power of the false agents in game  $G'$ ,

$$\begin{aligned}
\sum_{j=1}^k \varphi_{i_j}(G') &\leq \frac{P_i \cdot k \cdot (n+k-2)!}{(n-1)!(n+k-1)!} \\
&= \frac{P_i \cdot k}{(n-1)!(n+k-1)} \\
&= \frac{k}{(n-1)!(n+k-1)} \cdot n! \cdot \frac{P_i}{n!} \\
&= \frac{nk}{(n+k-1)} \varphi_i(G).
\end{aligned}$$

We now prove that this bound is asymptotically tight. Let  $G = [k, k, \dots, k; nk]$  be a unanimity WVG of  $n$  agents. It is clear that the only winning coalition consists of all the agents. So,  $\varphi_i(G) = \frac{1}{n}$  for all agents  $i \in I$  in the game. Suppose the last agent splits into  $k$  false identities each with weight 1, we have a new game  $G' = [k, k, \dots, k, \underbrace{1, 1, \dots, 1}_{k \text{ times}}; nk]$  of  $n+k-1$  agents. Then,

$$\varphi_i(G') = \frac{1}{n+k-1} \text{ for each agent } i \text{ in the altered game } G'. \text{ Hence, } \sum_{j=1}^k \varphi_{n_j}(G') = \frac{k}{n+k-1} = \frac{nk}{n+k-1} \varphi_n(G). \quad \square$$

**Theorem 2. (Lower Bound).** *Let  $G$  be a WVG of  $n$  agents. If an agent  $i$  alters  $G$  by splitting into  $k \geq 2$  false agents in a new game  $G'$ , then, the power index of the agent in the new game using the Shapley-Shubik index is at least  $\frac{k}{C(n+k-1, k-1)} \varphi_i(G)$ .*

<sup>1</sup> $C(n, r) = \frac{n!}{r!(n-r)!}$ .

<sup>2</sup>Consider a *director* who decides whether to take from  $\pi_r$  or  $\rho$ . He says “original” to take from  $\pi_r$  or “splinter” to take from  $\rho$ . He must say “original”  $n-1$  times and “splinter”  $k-1$  times.

*Proof.* Let agent  $i$  be a manipulator that splits into  $k$  false agents,  $i_1, \dots, i_k$ , in an altered game  $G'$ . Consider any  $i$ -pivotal-basis permutation  $\pi_r \in \Pi_{G-i}$  of agents in game  $G$ . Recall that  $r$  is defined such that when agent  $i$  is inserted into  $\pi_r$  at position  $r$ ,  $i$  is pivotal in game  $G$ . Now, consider a morphed permutation  $\pi'_r$  in game  $G'$  formed from inserting the false agents into  $\pi_r$  starting from position  $r$ , in which a false agent is pivotal. While there are many permutations  $\pi'_r$  that can be morphed from  $\pi_r$  in which a false agent is pivotal, at the very least, we know that if all the false agents are *adjacent* (in any order) and are inserted at position  $r$  in  $\pi_r$ , one of the false agents must be pivotal. Thus, for each  $\pi_r$ , there are  $k!$  such permutations that can be morphed from  $\pi_r$ . Notice, that we have ignored all other cases where the false agents are not adjacent in the permutation and one of the false agents is also pivotal.

Suppose there are  $P_i$   $i$ -pivotal permutations that can be formed from the set  $\Pi_{G-i}$  in game  $G$ , then, the sum of the Shapley-Shubik power of the false agents in game  $G'$ ,

$$\begin{aligned}
\sum_{j=1}^k \varphi_{i_j}(G') &\geq \frac{k! \cdot P_i}{(n+k-1)!} \\
&= \frac{k!}{(n+k-1)!} \cdot n! \cdot \frac{P_i}{n!} \\
&= \frac{k!n!}{(n+k-1)!} \varphi_i(G) \\
&= \frac{k}{C(n+k-1, k-1)} \varphi_i(G).
\end{aligned}$$

□

## 4 Banzhaf Power Index Bounds

**Theorem 3. (Upper Bound).** *Let  $G$  be a WVG of  $n$  agents. If an agent  $i$  alters  $G$  by splitting into  $k \geq 2$  false agents in a new game  $G'$ , then, the Banzhaf power index of the agent in the new game can be as much as  $k \cdot \beta_i(G)$ .*

*Proof.* Let  $i$  be a manipulator that splits into  $k$  false agents,  $i_1, \dots, i_k$ , with corresponding weights,  $w_{i_1}, \dots, w_{i_k}$ , in a new game  $G'$ . We assume without loss of generality that  $w_{i_1} \leq \dots \leq w_{i_k}$ . Define a *base coalition*,  $S_{G-i}$ , to be a set of agents from a winning coalition in a WVG  $G$  for which agent  $i$  is removed. Note that there are three possibilities for any agent to be critical in a winning coalition in game  $G$ :

1. Winning coalitions which do not contain agent  $i$ .
2. Winning coalitions which contain agent  $i$ , but in which  $i$  is not critical.
3. Winning coalitions in which agent  $i$  is critical.

Now, we need to transform each winning coalition in  $G$  to coalitions in the game  $G'$ , and then count the number of critical agents in each transformed coalition.

**Case 1:** Let  $X_1$  be the total number of winning coalitions in game  $G$  which do not contain agent  $i$ . Let  $X_2$  be the average number of the critical agents in each of these

winning coalitions. Since agent  $i$  is not present in any of these winning coalitions, the winning coalitions are not changed by transformation. Thus, the total number of critical agents from this case in the new game,  $G'$ , is  $X_1 \cdot X_2$ .

**Case 2:** Let  $Y_1$  be the total number of winning coalitions which contain agent  $i$ , but in which  $i$  is not critical. Let  $Y_2$  be the average number of critical agents in each of these coalitions. To create coalitions in  $G'$ , we add 1 or more of the false agents to the base coalition  $S_{G-i}$  to create a new winning coalition  $S'_{G'-i}$ . There are  $2^k - 1$  ways of selecting 1 or more of the false agents. No false agent will be critical, since agent  $i$  was not critical in  $S_{G-i}$ , but every critical agent in  $S_{G-i}$  will still be critical in  $S'_{G'-i}$ . Thus, we have a total of  $Y_1 \cdot Y_2 \cdot (2^k - 1)$  critical agents.

However, as we remove some of the false agents from the coalition, we could create new critical agents. For example, in the game  $[5,5,5;10]$ , coalition  $\{5,5,5\}$ , has no critical agents, but coalition  $\{5,5,4\}$  has two critical agents where none were critical before. Let  $Y_3$  be the average number of new critical agents created in each transformed coalition. Thus, the number of critical agents in this case can be counted as  $Y_1 \cdot Y_2 \cdot (2^k - 1) + Y_1 \cdot Y_3 \cdot (2^k - 2)$ .

**Case 3:** Let  $Z_1$  be the total number of winning coalitions in which agent  $i$  is critical. Let  $Z_2$  be the number of critical agents in each of these winning coalitions (not counting agent  $i$ ). Note that we do not expect  $Z_2$  to be the same for each winning coalition, but for simplicity, we assume  $Z_2$  is the average number. To create a coalition in  $G'$ , we add 1 or more of the false agents to the base coalition,  $S_{G-i}$ , to create a new winning coalition  $S'_{G'-i}$ . There are  $2^k - 1$  ways of selecting 1 or more of the false agents. Since  $i$  was critical in the original coalition, we must add enough false agents to  $S_{G-i}$  to make  $S'_{G'-i}$  winning. For example, if the sum of the weights of agents in  $S_{G-i}$  is  $w$ , and the quota of game  $G$  is  $q$ , the false agents which are added must be of a cumulative weight of at least  $q - w$ . We call this needed weight from the false agents, the  $i$ -need:

- If the sum of the weights of false agents added is less than  $i$ -need, the coalition is losing and no critical agents will be contributed from this case.
- If the sum of the weights of the false agents is as close to  $i$ -need without having excess false agents, the transformed coalition will be winning and every false agent will be critical. The critical agents of  $S_{G-i}$  will also be critical in  $S'_{G'-i}$ . For simplicity of analysis, we assume that the false agents are all of the same weight. Let  $p$  be the minimal number of false agents that are required to meet the  $i$ -need. There are  $C(k, p)$  ways of selecting which false agents are present in the transformed coalition. The number of winning coalitions in which the false agents are critical is  $Z_1 \cdot p \cdot C(k, p)$ , since there are  $Z_1$  base coalitions,  $C(k, p)$  ways of deciding which of the  $p$  false agents to include, and all the  $p$  false agents will be critical. However, in each of these transformed coalitions, the agents which were critical in the base coalition are

still critical. Thus, the total number of critical agents is:  $Z_1 \cdot p \cdot C(k, p) + Z_1 \cdot Z_2 \cdot C(k, p)$ .

- If the added false agents exceed  $i$ -need, the transformed coalition will be winning but it is possible that none of the false agents is critical. The critical agents of  $S_{G-i}$  will also be critical in  $S'_{G'-i}$ . We must count how many false agents are critical and how many total agents are critical from this case. For simplicity, we assume that all false agents are of the same weight, so that by adding an extra false agent, none of the false agents is critical. There are  $C(k, p+1) + \dots + C(k, k) = \sum_{j=p+1}^k C(k, j)$  ways of selecting  $p+1$  or more of the false agents. Thus, the total number of critical agents are  $Z_1 \cdot Z_2 \cdot \sum_{j=p+1}^k C(k, j)$ .

Putting it altogether, the total number of critical agents (including the  $k$  false agents):

$$X_1 \cdot X_2 + Y_1 \cdot Y_2 \cdot (2^k - 1) + Y_1 \cdot Y_3 \cdot (2^k - 2) + Z_1 \cdot p \cdot C(k, p) + Z_1 \cdot Z_2 \cdot C(k, p) + Z_1 \cdot Z_2 \cdot \sum_{j=p+1}^k C(k, j) = X_1 \cdot X_2 + Y_1 \cdot Y_2 \cdot (2^k - 1) + Y_1 \cdot Y_3 \cdot (2^k - 2) + Z_1 \cdot p \cdot C(k, p) + Z_1 \cdot Z_2 \cdot \sum_{j=p}^k C(k, j).$$

Now, the original Banzhaf power of agent  $i$  in  $G$ ,  $\beta_i(G) = \frac{Z_1}{Z_1 + X_1 \cdot X_2 + Y_1 \cdot Y_2 + Z_1 \cdot Z_2}$ . Similarly, the new power of agent  $i$  in game  $G'$  (which is the sum of the power of the false agents),  $\sum_{j=1}^k \beta_{i_j}(G') =$

$$\frac{Z_1 \cdot p \cdot C(k, p)}{X_1 \cdot X_2 + Y_1 \cdot Y_2 \cdot (2^k - 1) + Y_1 \cdot Y_3 \cdot (2^k - 2) + Z_1 \cdot p \cdot C(k, p) + Z_1 \cdot Z_2 \cdot \sum_{j=p}^k C(k, j)}.$$

The ratio,  $\tau = \frac{\sum_{j=1}^k \beta_{i_j}(G')}{\beta_i(G)}$ , gives,

$$\frac{p \cdot C(k, p) (Z_1 + X_1 \cdot X_2 + Y_1 \cdot Y_2 + Z_1 \cdot Z_2)}{X_1 \cdot X_2 + Y_1 \cdot Y_2 \cdot (2^k - 1) + Y_1 \cdot Y_3 \cdot (2^k - 2) + Z_1 \cdot p \cdot C(k, p) + Z_1 \cdot Z_2 \cdot \sum_{j=p}^k C(k, j)}.$$

Since we want to find the highest possible ratio, we need to determine cases which maximize the ratio. Note that the ratio of increase of  $power_i$  of agent  $i$  is bounded by,  $\frac{1}{power_i}$ , since the new power of the agent can be at most 1. We note that if  $p$  is small,  $\sum_{j=p}^k C(k, j)$  approaches  $2^k$ . Since a large denominator makes for a small ratio, terms in the denominator which are multiplied by  $2^k$  without terms in the numerator being multiplied by a large number drive down the ratio. Consider the case in which  $Y_1 \cdot Y_2 = 0$  and  $Y_1 \cdot Y_3 = 0$ . Let  $p = k$ . Now our ratio becomes:  $\frac{k(X_1 \cdot X_2 + Z_1 + Z_1 \cdot Z_2)}{X_1 \cdot X_2 + k \cdot Z_1 + Z_1 \cdot Z_2}$  which is bounded by  $k$ . Thus, there are cases where splitting into several false identities improves the power of a manipulator by a factor of as much as  $k$ .

We next show the existence of such a case. Let  $G = [w_1, w_2, \dots, w_n; q]$  be a unanimity WVG of  $n$  agents such that  $q = \sum_{i=1}^n w_i$ . It is clear that the only winning coalition consists of all the agents. So,  $\beta_i(G) = \frac{1}{n}$  for all agents  $i \in I$  in the game. Suppose the last agent splits into  $k$  false identities, we have a new game  $G' = [w_1, w_2, \dots, w_{n-1}, w_{n_1}, w_{n_2}, \dots, w_{n_k}; q]$  of  $n + k - 1$  agents. Then,  $\beta_i(G') = \frac{1}{n+k-1}$  for each agent  $i$  in the altered game  $G'$ . The ratio of the new power to the original power of the manipulator is  $\frac{n}{n+k-1}$ . So, as  $n$  goes to infinity, the denominator approaches  $n$ . Thus, the ratio goes to  $k$ . Similarly, as  $k$  goes to infinity, the denominator approaches  $k$ , and the ratio goes to  $n$ .  $\square$

**Theorem 4. (Lower Bound).** Let  $G = [w_1, \dots, w_n; q]$  be a WVG of  $n$  agents. If an agent  $i$  alters  $G$  by splitting into  $k \geq 2$  false agents in a new game  $G'$ , then, the Banzhaf power index of the agent in the new game is at least

$$\beta_i(G) \cdot \left[ \frac{1}{1 + \frac{(n-1) \cdot 2^{n+k-1}}{k \sum_{x \in I} \eta_x(G)}} \right].$$

*Proof.* Let agent  $i$  with weight  $w_i$  be a manipulator in a WVG  $G$  of  $n$  agents. Suppose  $i$  splits into  $k$  false agents,  $i_1, \dots, i_k$ , with corresponding weights,  $w_{i_1}, \dots, w_{i_k}$ , in a new game  $G'$ . We assume without loss of generality that  $w_{i_1} \leq \dots \leq w_{i_k}$ . Recall that  $\eta_i(G)$  is the number of winning coalitions for which an agent  $i$  is critical in a WVG  $G$ . We first bound the least number of winning coalitions in the altered game  $G'$  for which at least one of the false agents is critical. The idea is to consider winning coalitions in  $G'$  (having at least one false agent) that are derived from only the winning coalitions in  $G$  for which agent  $i$  is critical.

Define a *base coalition*,  $S_{G-i}$ , to be a set of agents from a winning coalition in a WVG  $G$  for which agent  $i$  is critical, but later removed, i.e.,  $w(S_{G-i}) < q$  and  $w(S_{G-i} \cup \{i\}) \geq q$ . It is clear that  $w(S_{G-i}) < q \leq w(S_{G-i}) + w_i = w(S_{G-i}) + w_{i_1} + \dots + w_{i_k}$ . The following are the possibilities for the relationships among the base coalition,  $S_{G-i}$ , the quota,  $q$ , and the weights,  $w_{i_1}, \dots, w_{i_k}$ , of the  $k$  false agents in the altered game  $G'$ :

0.  $q - w(S_{G-i}) \leq w_{i_1}$
1.  $w_{i_1} < q - w(S_{G-i}) \leq w_{i_2} \leq \dots \leq w_{i_k}$
2.  $w_{i_1} \leq w_{i_2} < q - w(S_{G-i}) \leq w_{i_3} \leq \dots \leq w_{i_k}$
- .
- .
- .
- k.  $q - w(S_{G-i}) > w_{i_k}$

We need to consider all the  $k+1$  cases above to determine which case gives the least number of winning coalitions in game  $G'$  for which at least one of the false agents is critical. Consider the extreme cases (i.e., cases 0 and  $k$ ) first;

**Case 0:**  $q - w(S_{G-i}) \leq w_{i_1}$ . In this case,  $i_1, i_2, \dots, i_k$ , are each critical for the coalitions,  $(S_{G-i} \cup \{i_1\}), (S_{G-i} \cup \{i_2\}), \dots, (S_{G-i} \cup \{i_k\})$ , respectively, in  $G'$ . Hence, the sum of the number of winning coalitions for the false agents,  $\sum_{j=1}^k \eta_{i_j}(G') = k \cdot \eta_i(G)$ .

**Case k:**  $q - w(S_{G-i}) > w_{i_k}$ . In this case, none of the false agents is critical on its own when it is included in  $S_{G-i}$ . The least possible number of coalitions in this case occurs when each of the false agents is critical in the winning coalition,  $(S_{G-i} \cup \{i_1, i_2, \dots, i_k\})$ . Hence, the sum of the number of winning coalitions for the false agents,  $\sum_{j=1}^k \eta_{i_j}(G') = k \cdot \eta_i(G)$ .

For the remaining cases (i.e., cases 1 to  $k-1$ ), we need to know the number of false agents that should be included in  $S_{G-i}$  such that the new coalition in  $G'$  is winning and at

least one of the false agents is critical in the coalition:

**Case 1:**  $w_{i_1} < q - w(S_{G-i}) \leq w_{i_2} \leq \dots \leq w_{i_k}$ . In this case,  $i_1$  is not critical in any coalition. However,  $i_2, i_3, \dots, i_k$ , are each critical, in the coalitions,  $(S_{G-i} \cup \{i_1, i_2\}), (S_{G-i} \cup \{i_1, i_3\}), \dots, (S_{G-i} \cup \{i_1, i_k\})$ , and the coalitions,  $(S_{G-i} \cup \{i_2\}), (S_{G-i} \cup \{i_3\}), \dots, (S_{G-i} \cup \{i_k\})$ , respectively. Thus, the sum of the number of winning coalitions for the false agents,  $\sum_{j=1}^k \eta_{i_j}(G') = 2(k-1) \cdot \eta_i(G)$ .

**Case 2:**  $w_{i_1} \leq w_{i_2} < q - w(S_{G-i}) \leq w_{i_3} \leq \dots \leq w_{i_k}$ . In this case, both  $i_1$  and  $i_2$  are critical in  $(S_{G-i} \cup \{i_1, i_2\})$ .  $i_3$  is critical for  $(S_{G-i} \cup \{i_3\}), (S_{G-i} \cup \{i_1, i_3\})$ , and  $(S_{G-i} \cup \{i_2, i_3\})$ . Similarly, by substituting each of the false agents,  $i_4, \dots, i_k$ , in place of  $i_3$ , in each of the last three coalitions, the false agents are each critical in the coalitions. Thus, the sum of the number of winning coalitions for the false agents,  $\sum_{j=1}^k \eta_{i_j}(G') = (2 + 3(k-2)) \cdot \eta_i(G) = (3k-4) \cdot \eta_i(G)$ .

In the same vein, it is easy to show that the least number of winning coalitions for the false agents in each of the remaining cases is at least  $k \cdot \eta_i(G)$ , for  $k \geq 2$ . Hence,

$$\sum_{j=1}^k \eta_{i_j}(G') \geq k \cdot \eta_i(G). \quad (3)$$

We now bound the number of winning coalitions in the altered game  $G'$  for which the non-manipulators are critical. Consider an arbitrary (non-manipulating) agent,  $j \neq i$ , in game  $G$ , and let

$$S_1 = \{S \subseteq I \setminus \{j\} : i \notin S, w(S) < q, w(S) + w_j \geq q\},$$

$$S_2 = \{S \subseteq I \setminus \{j\} : i \in S, w(S) < q, w(S) + w_j \geq q\}.$$

$S_1$  are winning coalitions which do not include the manipulator.  $S_2$  are winning coalitions which include the manipulator. We have that the number of winning coalitions in game  $G$  for which agent  $j$  is critical,  $\eta_j(G) = |S_1| + |S_2|$ . Now, all winning coalitions in  $G'$  will be derived from  $G$ . First, all winning coalitions in set  $S_1$  are also winning coalitions in  $G'$ . On the other hand, since  $i$  splits into  $k$  false agents in  $G'$ , winning coalitions for  $G'$  are derived from set  $S_2$  by replacing  $i$  with one or more of the false agents. Formally, we consider the following cases for the resultant winning coalitions for agent  $j$  in game  $G'$  when  $i$  splits into  $k$  false agents,  $i_1, \dots, i_k$ .

**Case A:** Let  $S \in S_1$ . Clearly, since  $i \notin S$ ,  $S$  remains unchanged from game  $G$  to  $G'$ . Hence, for this case, agent  $j$  remains critical in  $G'$  for  $|S_1|$  winning coalitions.

**Case B:** Let  $S \in S_2$ . Since  $i \in S$ , the winning coalitions in game  $G'$  for this case must include one or more of the  $k$  false agents. There are  $2^k - 1$  ways of selecting one or more of the false agents and adding them to  $S$ . Because the power of the false agents is the ratio of the number of winning coalitions the false agents are involved in divided by the total number of winning coalitions, the least power will be obtained when

the false agents are involved in the fewest winning coalitions and the non-manipulating agents are involved in the maximum number of winning coalitions. Thus, we seek to know the maximum number of coalitions in which the non-manipulating agents can participate. The case which yields the maximum number of winning coalitions is that agent  $j$  is critical for all coalitions (containing one or more of the false agents). Thus, agent  $j$  can be critical in at most  $(2^k - 1) \cdot |S_2|$  winning coalitions in game  $G'$ . Now, the number of winning coalitions in game  $G'$  for which agent  $j$  is critical,

$$\begin{aligned}\eta_j(G') &\leq |S_1| + (2^k - 1)|S_2| \\ &= |S_1| + |S_2| + (2^k - 2)|S_2| \\ &= \eta_j(G) + (2^k - 2)|S_2|\end{aligned}$$

It remains for us to bound  $|S_2|$ . We note, from elementary combinatorics<sup>3</sup> that, if a finite set  $A$  contains  $n$  distinct elements, a particular element  $a \in A$  occurs in exactly  $2^{n-1}$  subsets of the set  $A$ . Also, it is easy to see from the definition of  $S_2$  that  $|I \setminus \{j\}| = n - 1$ , and since the manipulating agent,  $i \in I \setminus \{j\}$ , then,  $|S_2| \leq 2^{n-2}$ . Hence,

$$\begin{aligned}\eta_j(G') &\leq \eta_j(G) + (2^k - 2) \cdot 2^{n-2} \\ &= \eta_j(G) + 2^{n+k-2} - 2^{n-1} \\ &\leq \eta_j(G) + 2^{n+k-2}.\end{aligned}$$

Thus, the total number of winning coalitions for all the  $n - 1$  non-manipulating agents in game  $G'$  is

$$\sum_{x \in I \setminus \{i\}} \eta_x(G') \leq \sum_{x \in I \setminus \{i\}} (\eta_x(G) + 2^{n+k-2}) \quad (4)$$

Putting it altogether, we substitute (3) and (4) in the sum of the Banzhaf index of the false agents in game  $G'$ :

$$\sum_{j=1}^k \beta_{i_j}(G') = \frac{\sum_{j=1}^k \eta_{i_j}(G')}{\sum_{j=1}^k \eta_{i_j}(G') + \sum_{x \in I \setminus \{i\}} \eta_x(G')} \quad (5)$$

$$\geq \frac{\sum_{j=1}^k \eta_{i_j}(G')}{\sum_{j=1}^k \eta_{i_j}(G') + \sum_{x \in I \setminus \{i\}} (\eta_x(G) + 2^{n+k-2})} \quad (6)$$

$$\geq \frac{k\eta_i(G)}{k\eta_i(G) + \sum_{x \in I \setminus \{i\}} (\eta_x(G) + 2^{n+k-2})} \quad (7)$$

$$= \frac{k\eta_i(G)}{k\eta_i(G) + \sum_{x \in I \setminus \{i\}} \eta_x(G) + \sum_{x \in I \setminus \{i\}} 2^{n+k-2}} \quad (8)$$

<sup>3</sup>The number of subsets in which any particular element of an  $n$ -set appears in its power set is just the counting of all the ways of selecting  $i$  out of the remaining  $n - 1$  elements of the set to be part of the subset which includes the element, with  $i = 0, 1, \dots, n - 1$ . This is given as  $\sum_{i=0}^{n-1} C(n - 1, i) = 2^{n-1}$ .

$$\geq \frac{k\eta_i(G)}{k \sum_{x \in I} \eta_x(G) + (n - 1) \cdot 2^{n+k-2}} \quad (9)$$

$$= \frac{k\eta_i(G)}{k \sum_{x \in I} \eta_x(G)} \cdot \left[ \frac{1}{1 + \frac{(n-1) \cdot 2^{n+k-2}}{k \sum_{x \in I} \eta_x(G)}} \right] \quad (10)$$

$$= \beta_i(G) \cdot \left[ \frac{1}{1 + \frac{(n-1) \cdot 2^{n+k-2}}{k \sum_{x \in I} \eta_x(G)}} \right]. \quad (11)$$

Note that inequality (6) holds since the denominator of the estimate (i.e., the right hand side of the inequality) is larger than that of the actual value. Similarly, (7) holds since the estimate uses a smaller number of winning coalitions for the false agents than that of the actual value. Clearly, if the same quantity is subtracted from the numerator and denominator, and the numerator is less than the denominator, then the ratio gets smaller. Finally, (9) is trivially true<sup>4</sup>.  $\square$

## 5 Conclusions

We consider false-name manipulation in WVGs when an agent splits into  $k > 2$  false identities. This problem, until now, has remained open for both the Shapley-Shubik and Banzhaf indices. We resolve this open problem. Specifically, we provide four non-trivial bounds when an agent splits into  $k \geq 2$  false agents using the two indices. One of the bounds is also shown to be asymptotically tight. The analyses of these novel results not only increase our understanding on the extent of power that manipulators may gain while they engage in false-name manipulation in WVGs, they also provide further insights into the problem which we believe may reveal methods on how to reduce the effects of the menace.

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<sup>4</sup>Let  $a, b, k \in \mathbb{N}$ , clearly,  $\frac{a}{ka+b} \geq \frac{a}{ka+kb}$ .