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Observations on the Minimality of Ranking Functions for Qualitative Conditional Knowledge Bases and Their Computation

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Abstract

Ordinal conditional functions (OCFs) provide a semantic domain for qualitative conditionals of the form "*if A, then (normally) B*" by ordering worlds according to their degree of surprise. Transferring the idea of maximum entropy to a more qualitative domain, c-representations of a knowledge base $\mathcal{R}$ consisting of a set of conditionals have been defined as OCFs satisfying in particular the property of conditional indifference. While c-representations for $\mathcal{R}$ can be specified as the solutions of a constraint satisfaction problem $CR(\mathcal{R})$, it has been an open problem whether there may be different minimal c-representations induced by minimal solutions of $CR(\mathcal{R})$. Another open question has been whether particular inequations in $CR(\mathcal{R})$ may be sharpened by transforming them into equations without loosing any minimal solutions, taking different notions of minimality into account. In this paper, we answer both questions and discuss further aspects of OCF minimality.

1 Introduction

Probably the most often used form of knowledge representation is some kind of *if-then* rules. In this paper, we consider knowledge bases $\mathcal{R}$ consisting of a set of qualitative conditionals that represent *if-then* rules allowing for exceptions, like *birds (normally) fly* or *computer scientists (normally) like their job*, formally denoted by $(\mathit{fly}|\mathit{bird})$ and $(\mathit{like\_job}|\mathit{computer\_scientist})$. Since such conditionals allow for exceptions, they require a powerful semantic domain. Ordinal conditional functions (OCFs) (Spohn 1988) order worlds according to their degree of implausibility (or surprise, respectively) and provide such a domain.

Different approaches have been proposed for determining ranking functions for a knowledge base $\mathcal{R}$, see e.g. (Goldszmidt, Morris, and Pearl 1993; Goldszmidt and Pearl 1996), (Goldszmidt, Morris, and Pearl 1993), (Weydert 1998), (Kern-Isberner 2001). In this paper we focus on c-representations (Kern-Isberner 2001; 2002) that are an extension of system Z (Goldszmidt, Morris, and Pearl 1993). C-representations for $\mathcal{R}$ can be obtained from the solutions of a constraint satisfaction problem $CR(\mathcal{R})$ that can be solved by constraint logic programming (Beierle, Kern-Isberner, and Södler 2013; 2012). We investigate several different notions of minimality, and show that there may be different non-equivalent c-representations induced by minimal solutions of $CR(\mathcal{R})$. Another open question we answer concerns the computation of minimal solutions when employing a seemingly obvious optimization by using a constraint satisfaction problem that is more restrictive than $CR(\mathcal{R})$, but that can be solved much more efficiently.

In Sec. 2, we briefly recall the background of conditional logic, OCFs, and c-representations. In Sec. 3, various notions of minimality are presented and illustrated. In Sec. 4, syntactic characteristics of a knowledge base are elaborated allowing for non-equivalent c-representations even in the case of ind-minimality, and Sec. 5 shows that an intuitive sharpening of $CR(\mathcal{R})$ may loose minimal solutions and may even classify non-minimal solutions as minimal ones. In Sec. 6 we conclude and point our further work.

2 Background

Conditional Logic and OCFs Let $\mathcal{L}$ be a propositional language over a finite set Σ of atoms $a, b, c, \ldots$. The formulas of $\mathcal{L}$ will be denoted by uppercase Roman letters $A, B, C, \ldots$. For conciseness of notation, we will omit the logical *and*-connective, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. Let Ω denote the set of possible worlds over $\mathcal{L}$; Ω will be taken simply as the set of all propositional interpretations over $\mathcal{L}$ and can be identified with the set of all complete conjunctions over Σ. For $\omega \in \Omega$, $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world ω.

By introducing a new binary operator $|$, we obtain the set $(\mathcal{L} \mid \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of *conditionals* over $\mathcal{L}$. $(B|A)$ formalizes "*if A then (normally) B*" and establishes a plausible, probable, possible etc connection between the *antecedent* A and the *consequence* B. Here, conditionals are supposed not to be nested, that is, antecedent and consequent of a conditional will be propositional formulas. A conditional $(B|A)$ is an object of a three-valued nature, partitioning the set of worlds Ω in three parts: those worlds satisfying AB, thus *verifying* the conditional, those worlds satisfying $A\overline{B}$, thus *falsifying* the conditional, and those worlds not fulfilling the premise A and so which the conditional may not be applied to at all. This allows us to associate to $(B|A)$ a *generalized indicator function* $\chi_{(B|A)}$ going back to (DeFinetti 1974) (where u stands for *unknown* or *indeter-*

minate):

$$\chi_{(B|A)}(\omega) = \begin{cases} 1 & \text{if} \quad \omega \models AB \\ 0 & \text{if} \quad \omega \models A\overline{B} \\ u & \text{if} \quad \omega \models \overline{A} \end{cases} \tag{1}$$

To give appropriate semantics to conditionals, they are usually considered within richer structures such as *epistemic states*. Besides certain (logical) knowledge, epistemic states also allow the representation of preferences, beliefs, assumptions of an intelligent agent. Basically, an epistemic state allows one to compare formulas or worlds with respect to plausibility, possibility, necessity, probability, etc.

Well-known ordinal approaches to represent epistemic states are Spohn's *ordinal conditional functions, OCFs*, (also called *ranking functions*) (Spohn 1988), and *possibility distributions* (Benferhat, Dubois, and Prade 1992), assigning degrees of plausibility, or of possibility, respectively, to formulas and possible worlds. In these frameworks, a conditional $(B|A)$ is valid (or *accepted*), if its confirmation, AB, is more plausible, possible, etc. than its refutation, $A\overline{B}$; a suitable degree of acceptance is calculated from the degrees associated with AB and $A\overline{B}$.

In this paper, we consider Spohn's OCFs (Spohn 1988). An OCF is a function $\kappa : \Omega \to \mathbb{N}$ expressing degrees of plausibility of propositional formulas where a higher degree denotes "less plausible" or "more surprising". At least one world must be regarded as being normal; therefore, $\kappa(\omega) = 0$ for at least one $\omega \in \Omega$. For expressing certain knowledge, the codomain of κ can be extended to $\mathbb{N} \cup \{\infty\}$. Each such ranking function can be taken as the representation of a full epistemic state of an agent. Each such κ uniquely extends to a function (also denoted by κ) mapping sentences and rules to $\mathbb{N} \cup \{\infty\}$ and being defined by

$$\kappa(A) \quad = \quad \begin{cases} \min\{\kappa(\omega) \mid \omega \models A\} & \text{if } A \text{ is satisfiable} \\ \infty & \text{otherwise} \end{cases} \tag{2}$$

for sentences $A \in \mathcal{L}$ and by

$$\kappa((B|A)) \quad = \quad \begin{cases} \kappa(AB) - \kappa(A) & \text{if } \kappa(A) \neq \infty \\ \infty & \text{otherwise} \end{cases} \tag{3}$$

for conditionals $(B|A) \in (\mathcal{L} \mid \mathcal{L})$. Note that $\kappa((B|A)) \geqslant 0$ since any ω satisfying AB also satisfies A and therefore $\kappa(AB) \geqslant \kappa(A)$. The belief of an agent being in epistemic state κ with respect to a default rule $(B|A)$ is determined by the satisfaction relation $\models_{\mathcal{O}}$ given by:

$$\kappa \models_{\mathcal{O}} (B|A) \quad \text{iff} \quad \kappa(AB) \ < \ \kappa(A\overline{B}) \tag{4}$$

Thus, $(B|A)$ is believed in κ iff the rank of AB (verifying the conditional) is strictly smaller than the rank of $A\overline{B}$ (falsifying the conditional). We say that κ *accepts* the conditional $(B|A)$ iff $\kappa \models_{\mathcal{O}} (B|A)$. Furthermore, κ accepts a knowledge base $\mathcal{R}$ iff it accepts every $R_i \in \mathcal{R}$; if there is no such κ, then $\mathcal{R}$ is *inconsistent*. For the rest of this paper, we assume that $\mathcal{R}$ is consistent.

C-Representations Different ways of determining a ranking function for a knowledge base $\mathcal{R}$ are given by *system Z* (Goldszmidt, Morris, and Pearl 1993; Goldszmidt and Pearl 1996) or its more sophisticated extension *system Z** (Goldszmidt, Morris, and Pearl 1993), see also (Bourne and Parsons 1999); for an approach using rational world rankings see (Weydert 1998). For quantitative knowledge bases of the form $\mathcal{R}_x = \{(B_1|A_1)[x_1], \ldots, (B_n|A_n)[x_n]\}$ with probability values x_i and with models being probability distributions P satisfying a probabilistic conditional $(B_i|A_i)[x_i]$ iff $P(B_i|A_i) = x_i$, a unique model can be chosen by employing the principle of maximum entropy (Paris 1994; Paris and Vencovska 1997; Kern-Isberner 1998); the maximum entropy model is a best model in the sense that it is the most unbiased one among all models satisfying $\mathcal{R}_x$.

Using the maximum entropy idea, in (Kern-Isberner 2002) a generalization of system Z* is suggested. Based on an algebraic treatment of conditionals, the notion of *conditional indifference* of κ with respect to $\mathcal{R}$ is defined and the following criterion for conditional indifference is given: An OCF κ is indifferent with respect to $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ iff $\kappa(A_i) < \infty$ for all $i \in \{1, \ldots, n\}$ and there are rational numbers $\kappa_0, \kappa_i^+, \kappa_i^- \in \mathbb{Q}$, $1 \leqslant i \leqslant n$, such that for all $\omega \in \Omega$,

$$\kappa(\omega) = \kappa_0 + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i B_i}} \kappa_i^+ + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \kappa_i^- . \tag{5}$$

When starting with an epistemic state of complete ignorance (i.e., each world ω has rank 0), for each rule $(B_i|A_i)$ the values κ_i^+, κ_i^- determine how the rank of each satisfying world and of each falsifying world, respectively, should be changed, taking (1) into account:

- If the world ω verifies the conditional $(B_i|A_i)$, – i.e., $\omega \models A_i B_i$ –, then κ_i^+ is used in the summation to obtain the value $\kappa(\omega)$.
- Likewise, if ω falsifies the conditional $(B_i|A_i)$, – i.e., $\omega \models A_i\overline{B_i}$ –, then κ_i^- is used in the summation instead.
- If the conditional $(B_i|A_i)$ is not applicable in ω, – i.e., $\omega \models \overline{A_i}$ –, then this conditional does not influence the value $\kappa(\omega)$.

κ_0 is a normalization constant ensuring that there is a smallest world rank 0. Employing the postulate that the ranks of a satisfying world should not be changed and requiring that changing the rank of a falsifying world may not result in an increase of the world's plausibility leads to the concept of a *c-representation*.

Definition 1 (c-representation (Kern-Isberner 2002)) *Any ranking function κ satisfying the conditional indifference condition (5) and $\kappa_i^+ = 0$, $\kappa_i^- \geqslant 0$ and*

$$\kappa(A_i B_i) < \kappa(A_i \overline{B_i}) \tag{6}$$

for $i = 1, \ldots, n$ is called a (special) c-representation *of $\mathcal{R}$.*

Note that for $i \in \{1, \ldots, n\}$, condition (6) expresses that κ accepts the conditional $R_i = (B_i|A_i) \in \mathcal{R}$ (cf. the definition of the satisfaction relation in (4)) and that this also implies $\kappa(A_i) < \infty$. Furthermore, $\kappa_0 = 0$ holds in Def. 1 since $\mathcal{R}$ is assumed to be consistent.

Thus, finding a c-representation for $\mathcal{R}$ amounts to choosing appropriate values $\kappa_1^-, \ldots, \kappa_n^-$. In (Beierle, Kern-

ω	$\kappa_1(\omega)$	$\kappa_2(\omega)$	$\kappa_3(\omega)$
fba	0	0	0
$fb\overline{a}$	1	1	1
$f\overline{b}a$	0	0	0
$f\overline{b}\overline{a}$	0	0	0

ω	$\kappa_1(\omega)$	$\kappa_2(\omega)$	$\kappa_3(\omega)$
$\overline{f}ba$	1	1	2
$\overline{f}b\overline{a}$	1	2	1
$\overline{f}\,\overline{b}a$	0	0	0
$\overline{f}\,\overline{b}\overline{a}$	0	0	0

Figure 1: Different ranking function κ_i accepting the rule set $\mathcal{R}_{birds}$ given in Example 1

Isberner, and Södler 2013) this situation is formulated as a constraint satisfaction problem $CR(\mathcal{R})$ whose solutions are vectors of the form $(\kappa_1^-, \ldots, \kappa_n^-)$ determining c-representations of $\mathcal{R}$. The formulation of $CR(\mathcal{R})$ requires that the κ_i^- are natural numbers (and not just rational numbers) and that $\min(\emptyset) = \infty$.

Definition 2 ***[$CR(\mathcal{R})$ (Beierle, Kern-Isberner, and Södler 2013)]*** *Let $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$. The constraint satisfaction problem for c-representations of $\mathcal{R}$, denoted by $CR(\mathcal{R})$, is given by the conjunction of the constraints, for all $i \in \{1, \ldots, n\}$:*

$$\kappa_i^- \geqslant 0 \tag{7}$$

$$\kappa_i^- > \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- \tag{8}$$

A solution of $CR(\mathcal{R})$ is an n-tuple $(\kappa_1^-, \ldots, \kappa_n^-)$ of natural numbers, and with $Sol_{CR}(\mathcal{R})$ we denote the set of all solutions of $CR(\mathcal{R})$.

Proposition 1 ***(Beierle, Kern-Isberner, and Södler 2013)*** *For $\mathcal{R} = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ let $\vec{\kappa} = (\kappa_1^-, \ldots, \kappa_n^-) \in Sol_{CR}(\mathcal{R})$. Then the function κ defined by*

$$\kappa(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \kappa_i^- \tag{9}$$

and denoted by $\kappa_{\vec{\kappa}}$, is an OCF that accepts $\mathcal{R}$.

Given a knowledge base $\mathcal{R} = \{R_1, \ldots, R_n\}$ of conditionals, a ranking function κ accepting every R_i represents an epistemic state of an agent accepting $\mathcal{R}$. Every OCF κ accepting $\mathcal{R}$ inductively completes the knowledge given by $\mathcal{R}$, and for any consistent $\mathcal{R}$ there may be many different such κ, each representing a complete set of beliefs with respect to every possible formula A and every conditional $(B|A)$.

Example 1 *Let $\mathcal{R}_{birds} = \{R_1, R_2, R_3\}$ be given by:*

$R_1:$	$(f\|b)$	<u>b</u>irds <u>f</u>ly
$R_2:$	$(a\|b)$	<u>b</u>irds are <u>a</u>nimals
$R_3:$	$(a\|fb)$	<u>f</u>lying <u>b</u>irds are <u>a</u>nimals

In Figure 1, three different OCFs κ_1, κ_2, κ_3 accepting $\mathcal{R}_{birds}$ are given. Thus, for any $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3\}$ it holds that $\kappa_i \models_{\mathcal{O}} R_j$. In order to illustrate the evaluation of beliefs, consider the conditional $(a|b\overline{f})$ ("Are non-flying birds animals?"*) that is not contained in $\mathcal{R}$. For κ_3, we get $\kappa_3(ab\overline{f}) = 2$ and $\kappa_3(\overline{a}b\overline{f}) = 1$ and therefore $\kappa_3 \not\models_{\mathcal{O}} (a|b\overline{f})$ so that the conditional $(a|b\overline{f})$ is not accepted by κ_3. On the other hand, for κ_2 we get $\kappa_2(ab\overline{f}) = 1$ and $\kappa_2(\overline{a}b\overline{f}) = 2$ and therefore $\kappa_2 \models_{\mathcal{O}} (a|b\overline{f})$.*

The full beliefs about non-flying birds being animals or not, represented by the conditionals $(a|b\overline{f})$ and $(\overline{a}|b\overline{f})$, are given by the following table:

$\kappa_1 \not\models_{\mathcal{O}} (a\|b\overline{f})$	$\kappa_2 \models_{\mathcal{O}} (a\|b\overline{f})$	$\kappa_3 \not\models_{\mathcal{O}} (a\|b\overline{f})$
$\kappa_1 \not\models_{\mathcal{O}} (\overline{a}\|b\overline{f})$	$\kappa_2 \not\models_{\mathcal{O}} (\overline{a}\|b\overline{f})$	$\kappa_3 \models_{\mathcal{O}} (\overline{a}\|b\overline{f})$

An agent being in epistemic state κ_2 believes that non-flying birds are animals and does not believe that non-flying birds are not animals. An agent being in epistemic state κ_3 does not believe that non-flying birds are animals and believes that non-flying birds are not animals. An agent being in epistemic state κ_1 is completely indifferent with respect to non-flying birds being animals or not, since she considers a world where non-flying birds are animals as equally plausible (or equally surprising) as a world where non-flying birds are not animals.

The computation of OCFs being c-representations by solving the constraint satisfaction problem $CR(\mathcal{R})$ is illustrated in the following example.

Example 2 *Let $\mathcal{R}_{birds} = \{R_1, R_2, R_3\}$ be as in Example 1. From (8) we get*

$$\kappa_1^- > 0, \quad \kappa_2^- > 0 - min\{\kappa_1^-, \kappa_3^-\}, \quad \kappa_3^- > 0 - \kappa_2^-$$

and since $\kappa_i^- \geqslant 0$ according to (7), the two vectors

$$sol_1 = (\kappa_1^-, \kappa_2^-, \kappa_3^-) = (1, 0, 1)$$
$$sol_2 = (\kappa_1^-, \kappa_2^-, \kappa_3^-) = (1, 1, 0)$$

are two different solutions of $CR(\mathcal{R}_{birds})$. The OCF κ_{sol_1} (resp. κ_{sol_2}) induced by sol_1 (resp. sol_2) according to (9) is κ_1 (resp. κ_2) as given in Example 1.

3 Notions of Minimality

Example 1 illustrates that there are different ways of completing the knowledge given by a conditional knowledge base $\mathcal{R}$. While in principle, one is interested in characterizing and determining the full set of accepting OCFs, it is a crucial question whether some κ is to be preferred to some other κ', and whether among the preferred ones there is a unique "best" κ. Among all OCFs accepting $\mathcal{R}$, usually a less specific OCF – i.e., regarding worlds as plausible as possible while taking $\mathcal{R}$ into account – is preferred over a more specific one. Thus, one is typically interested in *minimal* solutions. In (Goldszmidt and Pearl 1996), an OCF κ accepting $\mathcal{R}$ is said to be minimal iff for every other κ' accepting $\mathcal{R}$ there exists a world $\omega \in \Omega$ with $\kappa(\omega) < \kappa'(\omega)$. In (Bourne 1999; Bourne and Parsons 1999), minimality is defined with respect to vectors inducing ranking functions, and an algorithm for finding a minimal solution is given; there may be more than one minimal solutions, but the algorithm fails to find more than one minimal one.

Since in this paper, our focus is on c-representations, and since for any $\mathcal{R}$, the OCFs being c-representations and accepting $\mathcal{R}$ are induced by the solutions of $CR(\mathcal{R})$, we will consider different orderings on $Sol_{CR}(\mathcal{R})$, leading to three different minimality notions. A complete ordering on $Sol_{CR}(\mathcal{R})$ is obtained by using the sum of the κ_i^-, i.e.,

$$(\kappa_1^-, \dots, \kappa_n^-) \preccurlyeq_+ (\kappa'^-_1, \dots, \kappa'^-_n) \quad \text{iff} \quad \sum_{1 \leqslant i \leqslant n} \kappa_i^- \leqslant \sum_{1 \leqslant i \leqslant n} \kappa'^-_i. \tag{10}$$

A vector $\vec{\kappa} \in Sol_{CR}(\mathcal{R})$ is *sum-minimal* iff $\vec{\kappa} \preccurlyeq_+ \vec{\kappa}'$ for all $\vec{\kappa}' \in Sol_{CR}(\mathcal{R})$.

The component-wise ordering $\preccurlyeq_{cw}$ is defined by

$$(\kappa_1^-, \dots, \kappa_n^-) \preccurlyeq_{cw} (\kappa'^-_1, \dots, \kappa'^-_n) \quad \text{iff} \quad \kappa_i^- \leqslant \kappa'^-_i \text{ for all } i \in \{1, \dots, n\}. \tag{11}$$

Since $\preccurlyeq_{cw}$ is reflexive, transitive, and antisymmetric, it yields a partial order $\preccurlyeq_{cw}$ on $Sol_{CR}(\mathcal{R})$. A vector $\vec{\kappa}$ is *cw-minimal* iff there is no vector $\vec{\kappa}' \in Sol_{CR}(\mathcal{R})$ such that $\vec{\kappa}' \preccurlyeq_{cw} \vec{\kappa}$ and $\vec{\kappa} \not\preccurlyeq_{cw} \vec{\kappa}'$.

Still another alternative is to compare the full OCFs $\kappa_{\vec{\kappa}}$ induced by $\vec{\kappa} = (\kappa_1^-, \dots, \kappa_n^-)$ according to (9), yielding the partial ordering $\preccurlyeq_O$ on $Sol_{CR}(\mathcal{R})$ defined by:

$$(\kappa_1^-, \dots, \kappa_n^-) \preccurlyeq_O (\kappa'^-_1, \dots, \kappa'^-_n) \quad \text{iff} \quad \kappa_{\vec{\kappa}}(\omega) \leqslant \kappa_{\vec{\kappa}'}(\omega) \text{ for all } \omega \in \Omega. \tag{12}$$

The relation $\preccurlyeq_O$ is reflexive and transitive; however, since different vectors may induce the same OCF, it is not antisymmetric. Thus, $\preccurlyeq_O$ defines a partial preorder on $Sol_{CR}(\mathcal{R})$. A vector $\vec{\kappa}$ is *minimal with respect to the induced OCF* (or just *ind-minimal*) iff there is no vector $\vec{\kappa}' \in Sol_{CR}(\mathcal{R})$ such that $\vec{\kappa}' \preccurlyeq_O \vec{\kappa}$ and $\vec{\kappa} \not\preccurlyeq_O \vec{\kappa}'$. Two ind-minimal solution vectors $\vec{\kappa}, \vec{\kappa}'$ are *equivalent* iff $\kappa_{\vec{\kappa}} = \kappa_{\vec{\kappa}'}$. For instance, the two solution vectors sol_1 and sol_2 in Example 2 are both sum-minimal and also cw-minimal, but only sol_1 is ind-minimal.

4 Non-Equivalent Ind-Minimal Solutions

In (Beierle, Kern-Isberner, and Södler 2013; 2012) a software implementation `GenOCF` of a solver for $CR(\mathcal{R})$ using constraint logic programming is presented. For each of the three orderings $\preccurlyeq_+$, $\preccurlyeq_{cw}$, and $\preccurlyeq_O$, a corresponding mode of `GenOCF` computes all minimal solutions of $CR(\mathcal{R})$. There are many examples demonstrating that two sum-minimal solutions of $CR(\mathcal{R})$ may induce different OCFs; the same holds also for two different cw-minimal solutions (cf. Example 2). On the other hand, in none of the examples of knowledge bases $\mathcal{R}$ evaluated in (Beierle, Kern-Isberner, and Södler 2013; 2012), multiple ind-minimal solutions generating different OCFs were found, and so far, it has been unclear whether this is a general property of c-representations and thus of the corresponding constraint satisfaction problem $CR(\mathcal{R})$. In the following, we investigate and elaborate syntactic characteristics of $\mathcal{R}$ and show that for various syntactic variations, different ind-minimal solutions exist that lead to different induced OCFs.

Transitive Connections We call a conditional $(C|A)$ *transitive connection* of $(C|B)$ and $(B|A)$. The following example provides a knowledge base containing a transitive connection.

Example 3 *The following table presents the situation of a knowledge base $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$ with a transitive connection. Verifying and falsifying worlds are indicated by v and f, respectively. $CR(\mathcal{R})$ contains the constraints*

$$\kappa_1^- > 0 - 0, \qquad \kappa_2^- > 0 - \kappa_4^-,$$
$$\kappa_3^- > 0 - min\{\kappa_2^- + \kappa_4^-, \kappa_1^-\}, \qquad \kappa_4^- > 0 - \kappa_2^-$$

and there are two ind-minimal solution vectors sol_1 and sol_2; both induce the same OCF, i.e., $\kappa_{sol_1} = \kappa_{sol_2}$.

ω	R_1: $(b\|a)$	R_2: $(c\|b)$	R_3: $(c\|a)$	R_4: $(\overline{b}\|\overline{c})$	$\kappa_{sol_1}(\omega)$	$\kappa_{sol_2}(\omega)$
abc	v	v	v	$-$	0	0
$ab\overline{c}$	v	f	f	f	1	1
$a\overline{b}c$	f	$-$	v	$-$	1	1
$a\overline{b}\overline{c}$	f	$-$	f	v	1	1
$\overline{a}bc$	$-$	v	$-$	$-$	0	0
$\overline{a}b\overline{c}$	$-$	f	$-$	f	1	1
$\overline{a}\overline{b}c$	$-$	$-$	$-$	$-$	0	0
$\overline{a}\overline{b}\overline{c}$	$-$	$-$	$-$	v	0	0
sol_1	1	1	0	0		
sol_2	1	0	0	1		

Thus, in Example 3, we have two different ind-minimal solutions $sol_1, sol_2 \in Sol_{CR}(\mathcal{R})$. Since they both induce the same OCF, we have $sol_1 \preccurlyeq_O sol_2$ and $sol_2 \preccurlyeq_O sol_1$.

Replacing $(\overline{b}|\overline{c})$ in Example 3 by $(\overline{b}|\overline{a}\,\overline{c})$ changes the situation: sol_1 is still a solution, but it is now the unique ind-minimal solution. However, when replacing $(\overline{b}|\overline{c})$ by $(\overline{b}|\overline{a})$, another new situation arises.

Example 4 *Replacing $(\overline{b}|\overline{c})$ by $(\overline{b}|\overline{a})$ in Example 3 yields*

ω	R_1: $(b\|a)$	R_2: $(c\|b)$	R_3: $(c\|a)$	R_4: $(\overline{b}\|\overline{a})$	$\kappa_{sol_1}(\omega)$	$\kappa_{sol_2}(\omega)$
abc	v	v	v	$-$	0	0
$ab\overline{c}$	v	f	f	$-$	1	1
$a\overline{b}c$	f	$-$	v	$-$	1	1
$a\overline{b}\overline{c}$	f	$-$	f	$-$	1	2
$\overline{a}bc$	$-$	v	$-$	f	1	1
$\overline{a}b\overline{c}$	$-$	f	$-$	f	2	1
$\overline{a}\overline{b}c$	$-$	$-$	$-$	v	0	0
$\overline{a}\overline{b}\overline{c}$	$-$	$-$	$-$	v	0	0
sol_1	1	1	0	1		
sol_2	1	0	1	1		

where $CR(\mathcal{R})$ now contains the constraints:

$$\kappa_1^- > 0 - 0, \qquad \kappa_2^- > 0 - min\{\kappa_3^-, \kappa_4^-\},$$
$$\kappa_3^- > 0 - min\{\kappa_1^-, \kappa_2^-\}, \qquad \kappa_4^- > 0 - 0.$$

Note that in Example 4, we have again two ind-minimal solutions $sol_1, sol_2 \in Sol_{CR}(\mathcal{R})$, but this time $\kappa_{sol_1} \neq \kappa_{sol_2}$, and neither $sol_1 \preccurlyeq_O sol_2$ nor $sol_2 \preccurlyeq_O sol_1$ holds. Thus, transitive connections in a knowledge base may lead to non-equivalent ind-minimal solutions.

Other Situations with Multiple Ind-Minimal Solutions We call two conditionals of the form $(B|A)$ and $(B|\overline{A})$ *antecedent complementary*. Also if a knowledge base contains antecedent complementary conditionals, there may be non-equivalent ind-minimal solutions.

Example 5 *Antecedent complementary conditionals:*

ω	R_1: $(b\|a)$	R_2: $(b\|\overline{a})$	R_3: $(a\|c)$	R_4: $(b\|\overline{c})$	$\kappa_{sol_1}(\omega)$	$\kappa_{sol_2}(\omega)$
abc	v	$-$	v	$-$	0	0
$ab\overline{c}$	v	$-$	$-$	v	0	0
$a\overline{b}c$	f	$-$	v	$-$	1	1
$a\overline{b}\overline{c}$	f	$-$	$-$	f	1	2
$\overline{a}bc$	$-$	v	f	$-$	1	1
$\overline{a}b\overline{c}$	$-$	v	$-$	v	0	0
$\overline{a}\overline{b}c$	$-$	f	f	$-$	2	1
$\overline{a}\overline{b}\overline{c}$	$-$	f	$-$	f	1	1
sol_1	1	1	1	0		
sol_2	1	0	1	1		

Here, $CR(\mathcal{R})$ contains the constraints:

$$\kappa_1^- > 0 - 0, \quad \kappa_2^- > 0 - min\{\kappa_3^-, \kappa_4^-\},$$
$$\kappa_3^- > 0 - 0, \quad \kappa_4^- > 0 - min\{\kappa_1^-, \kappa_2^-\}.$$

Thus, as in Example 4, also in Example 5 there are two ind-minimal solutions sol_1, sol_2 with $\kappa_{sol_1} \neq \kappa_{sol_2}$.

Based on the findings presented above regarding the existence of multiple ind-minimal ranking functions, we also investigated subconditionals and perpendicular conditionals (Kern-Isberner 2001). Both for knowledge bases containing subconditionals and for knowledge bases containing perpendicular conditionals we were able to construct examples with different ind-minimal OCFs. However, in these cases the knowledge bases exhibiting this behaviour were slightly more complex, having either more conditionals or using more propositional variables; it is still an open question whether this is a general property, or whether also corresponding examples with just four conditionals over only three propositional variables exist. We are grateful to an anonymous reviewer for referring to the *nested crossing* example in (Weydert 2003, p. 296); in our framework of employing c-representations, also this example yields two different ind-minimal solutions.

5 Sharpening Inequations to Equations

The constraints in $CR(\mathcal{R})$ given by (8) ensure that each conditional $(B_i|A_i) \in \mathcal{R}$ is accepted. Since all κ_i^- are assumed to be natural numbers, we can replace the strict inequation (8) by

$$\kappa_i^- \geqslant 1 + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- \quad (13)$$

without changing the set of solutions $Sol_{CR}(\mathcal{R})$. As one is interested in minimal solutions and thus in minimizing the values of all κ_i^-, one could be tempted to replace the non-strict inequality in (13) by an equality as in:

$$\kappa_i^- = 1 + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- - \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- \quad (14)$$

However, as pointed out in (Beierle, Kern-Isberner, and Södler 2012), using just (14) instead of (8), one might loose a solution in the case where the right hand side of the inequation (8) is negative since then (14) might require that κ_i^- is negative, which is inconsistent with (7). If the right hand side of (8) is negative, $\kappa_i^- \geqslant 0$ due to (7) already ensures that (8) holds, so in that case no additional requirement on κ_i^- is needed. Thus, (14) should be used only if the right hand side of (8) is not negative, i.e. if

$$1 + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- > \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- \quad (15)$$

holds (Beierle, Kern-Isberner, and Södler 2012). Putting these constraints together yields

$$\kappa_i^- = \quad 1 + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^- \quad (16)$$
$$- \min\{1 + \min_{\omega \models A_i B_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^-, \ \min_{\omega \models A_i \overline{B}_i} \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B_j}}} \kappa_j^-\}$$

Definition 3 ($CRE(\mathcal{R})$) *$CRE(\mathcal{R})$ is the constraint system obtained from $CR(\mathcal{R})$ by replacing (8) by (16).*

Note that just as required, (16) reduces to $\kappa_i^- = 0$ if (15) does not hold. As an optimization of $CR(\mathcal{R})$, in (Beierle, Kern-Isberner, and Södler 2012) is suggested to use $CRE(\mathcal{R})$ instead of $CR(\mathcal{R})$. Note that this transforms a strictly-greater-than relationship into an equation; thus it should be clearly distinguished from the modelling of a constraint $x > y$ by $x \geqslant y + 1$ which might be done by the underlying constraint solver. For all knowledge bases investigated previously and for different notions of minimality, this sharpening of $CR(\mathcal{R})$ to $CRE(\mathcal{R})$ did not loose any minimal solution; on the other hand, the runtime needed for solving $CRE(\mathcal{R})$ is significantly smaller (Beierle, Kern-Isberner, and Södler 2012).

However, in the following, we will show that there are knowledge bases where using $CRE(\mathcal{R})$ instead of $CR(\mathcal{R})$ does loose minimal solutions, and that this is the case for any of the three notions of minimality considered above.

Example 6 ($CR(\mathcal{R})$) *The following table presents the situation of a knowledge base $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$ in a representation as used above.*

ω	R_1: $(b\|a)$	R_2: $(b\|\overline{a})$	R_3: $(a\|c)$	R_4: $(c\|\overline{b})$	$\kappa_{sol_1}(\omega)$	$\kappa_{sol_2}(\omega)$
abc	v	$-$	v	$-$	0	0
$ab\overline{c}$	v	$-$	$-$	$-$	0	0
$a\overline{b}c$	f	$-$	v	v	1	1
$a\overline{b}\overline{c}$	f	$-$	$-$	f	2	3
$\overline{a}bc$	$-$	v	f	$-$	1	1
$\overline{a}b\overline{c}$	$-$	v	$-$	$-$	0	0
$\overline{a}\overline{b}c$	$-$	f	f	v	2	1
$\overline{a}\overline{b}\overline{c}$	$-$	f	$-$	f	2	2
sol_1	1	1	1	1		
sol_2	1	0	1	2		

There are two solutions sol_1 and sol_2 of $CR(\mathcal{R})$; note that both solutions are sum-minimal, cw-minimal, and also ind-minimal.

We will now investigate $CR(\mathcal{R})$ for $\mathcal{R}$ from Example 6.

Example 7 ($\boldsymbol{CRE(\mathcal{R})}$) *In Example 6, $CR(\mathcal{R})$ contains the following inequations:*

$$\kappa_1^- > 0 - 0$$
$$\kappa_2^- > 0 - \min\{\kappa_3^-, \kappa_4^-\}$$
$$\kappa_3^- > 0 - 0$$
$$\kappa_4^- > \min\{\kappa_1^-, \kappa_2^- + \kappa_3^-\} - \min\{\kappa_1^-, \kappa_2^-\}$$

Instead of these four inequations, $CRE(\mathcal{R})$ contains the following four equations:

$$\kappa_1^- = 1 + 0 - 0 = 1$$
$$\kappa_2^- = 1 + 0 - \min\{\min\{\kappa_3^-, \kappa_4^-\}, 1 + 0\}$$
$$\kappa_3^- = 1 + 0 - 0 = 1$$
$$\kappa_4^- = 1 + \min\{\kappa_1^-, \kappa_2^- + \kappa_3^-\} - \min\{\min\{\kappa_1^-, \kappa_2^-\}, 1 + \min\{\kappa_1^-, \kappa_2^- + \kappa_3^-\}\}$$

Solving $CRE(\mathcal{R})$ reveals that the solution $sol_2 = (1,0,1,2)$ of $CR(\mathcal{R})$ is also a solution of $CRE(\mathcal{R})$. However, the other solution $sol_1 = (1,1,1,1)$ of $CR(\mathcal{R})$ is not a solution of $CRE(\mathcal{R})$ since the second equation of $CRE(\mathcal{R})$ given above does not hold.

Thus, we have shown that in general, when using $CRE(\mathcal{R})$ instead of $CR(\mathcal{R})$, one might loose minimal solutions, and this observation holds for any of the three notions of minimality considered above. Replacing R_3 and R_4 in Example 6 by $(a|\overline{b}\overline{c})$ and $(\overline{c}|\overline{b})$ yields $\mathcal{R}'$ where we even get an incorrect answer since the unique ind-minimal solution of $CR(\mathcal{R}')$ is not among the solutions of $CRE(\mathcal{R}')$.

6 Conclusion and Further Work

Ordinal conditional functions are a powerful means for representing the semantics of qualitative conditional knowledge bases. In this paper, we focused on c-representations and, using various notions of minimality, studied questions regarding the existence of non-equivalent minimal solutions and regarding the computation of c-representations by solving a constraint satisfaction problem.

While preferring smaller ranking functions over larger ones seems reasonable in many cases, it is still an open problem how to determine in which cases a minimal solution is also considered to be a best solution. For instance, taking into account the suggested interpretation of the birds' world in Example 1, the epistemic state κ_2 might be preferred to κ_1 (although κ_1 is the unique minimal model) since only in κ_2 an agent believes that birds are still animals even if they can not fly. Of course, this background knowledge is not present in the simple knowledge base $\mathcal{R}_{birds}$, and there are other interpretations of a, b, f where κ_1 might be preferred to κ_2. Further research is needed how to specify such kind of background knowledge and how to take it into account for specifying and finding minimal and best solutions.

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