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Representation of Indifference and Preference Ordering Between Random Variables by Coherent Upper and Lower Conditional Previsions Defined by Hausdorff Outer and Inner Measures

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Abstract

Coherent lower and upper conditional previsions defined by Hausdorff inner and outer measures are proposed to represent respectively a partial strict order and a complete indifference relations between random variables. The two binary relations can describe the activity of the conscious human thought ruled by the antisymmetric property and the unconscious human thought which is governed by the symmetric principle and the generalization principle.

Introduction

Because of incomplete and inaccurate information, a measure of uncertainty can be represented by coherent imprecise probabilities (Walley 1991), which consist of a class of probability measures and not by a single probability measure. Coherent upper and lower probability are respectively the maximum and the minimum of the given class. The extensions to the class of all bounded random are called coherent upper and lower previsions. A priori measure of uncertainty is the level of knowledge each subject has before having a new piece of information, denoted by the set B; the measure of uncertainty that quantifies the level of knowledge each subject has on a posteriori situations is a coherent upper probability conditioned to the state B. A new model of coherent conditional upper probabilities defined by Hausdorff outer measures has been proposed (Doria 2007), (Doria 2012), (Doria 2015), (Doria 2019), to affirm that there is updating of knowledge if the a priori and a posteriori measures of uncertainty are different, that is if they are defined by different Hausdorff outer measures. It occurs when the new piece of information represented by a set, has a different complexity, measured in terms of Hausdorff dimension of the set, with respect to the previous information. The new model of coherent upper conditional probabilities based on Hausdorff outer measures has been proposed because coherent upper and lower conditional probabilities cannot be obtained as extensions of linear conditional probability defined by the Radon-Nikodym derivative, as in the axiomatic approach ((Billingsley 1986)); it occurs because one of the defining properties of the Radon-Nikodym derivative, that is to be measurable with respect to the σ-field of the conditioning events, contradicts a necessary condition for the coherence (Doria 2012).

Many properties of the coherent lower previsions can be obtained by the conjugate coherent upper conditional previsions but the two non-linear functionals represent different binary relations between random variables since preference orderings represented by the coherent lower previsions satisfy the antisymmetric property which is not satisfied by the binary relation represented by their conjugate coherent upper conditional previsions (Doria 2015), (Doria 2019).

In this paper links between the new model of uncertainty representation and the brain's activity are investigated.

Coherent upper conditional previsions defined by Hausdorff outer measures

Let (Ω, d) be a metric space and let **B** be a partition of Ω.

A bounded random variable is a function $X : \Omega \to \Re$ and $L(\Omega)$ is the class of all bounded random variables defined on Ω; for every $B \in \mathbf{B}$ denote by $X|B$ the restriction of X to B and by sup$(X|B)$ the supremum value that X assumes on B. Let $L(B)$ be the class of all bounded random variables $X|B$. Denote by I_A the indicator function of any event $A \in \wp(B)$, i.e. $I_A(\omega) = 1$ if $\omega \in A$ and $I_A(\omega) = 0$ if $\omega \in A^c$.

For every $B \in \mathbf{B}$ coherent upper conditional previsions $\overline{P}(\cdot|B)$ are functionals defined on $L(B)$ (Walley 1991).

Definition 1 *Coherent upper conditional previsions are functionals $\overline{P}(\cdot|B)$ defined on $L(B)$, such that the following conditions hold for every X and Y in $L(B)$ and every strictly positive constant λ:*

1) $\overline{P}(X|B) \le \sup(X|B)$*;*

2) $\overline{P}(\lambda X|B) = \lambda\overline{P}(X|B)$ *(positive homogeneity);*

3) $\overline{P}(X+Y)|B) \le \overline{P}(X|B) + \overline{P}(Y|B)$ *(subadditivity);*

4) $\overline{P}(I_B|B) = 1$.

$1) - 4)$ in Definition 1 is said to be axioms coherence.

Suppose that $\overline{P}(X|B)$ is a coherent upper conditional prevision on $L(B)$ then its conjugate coherent lower conditional prevision is defined by $\underline{P}(X|B) = -\overline{P}(-X|B)$. Let

K be a linear space contained in $L(B)$; if for every X belonging to K we have $P(X|B) = \underline{P}(X|B) = \overline{P}(X|B)$ then $P(X|B)$ is called a coherent *linear* conditional prevision (de Finetti 1970), (de Finetti 1974),(Regazzini 1987) and it is a linear, positive and positively homogenous functional on $L(B)$.

The unconditional coherent upper prevision $\overline{P} = \overline{P}(\cdot|\Omega)$ is obtained as a particular case when the conditioning event is Ω. Coherent upper conditional probabilities are obtained when only 0-1 valued random variables are considered.

An upper prevision is a real-valued function defined on some class of bounded random variables K. A necessary and sufficient condition for an upper prevision $\overline{P}$ to be coherent is to be the *upper envelope* of linear previsions, i.e. there is a class M of linear previsions such that $\overline{P}$ =sup$\{P : P \in M\}$.

A new model of coherent upper conditional probability based on Hausdorff outer measures is introduced in (Doria 2007),(Doria 2012), (Doria 2015). For the definition of Hausdorff outer measure and its basic properties see (Rogers 1970) and (Falconer 1986).

The model is a generalization of Bayes Theorem, used to update probabilities when a new piece of information, represented by the event B, is acquired.

The innovative aspect consists in the fact that the measure that is used to define the conditional upper probability depends on the complexity of the conditioning event, given in terms of Hausdorff dimension of the set B.

Therefore the events with a zero-value a priori probability determine the change of the measure of uncertainty that represents the level of knowledge of the subject.

Let (Ω, d) be a metric space and let **B** be partition of Ω.

Let $\delta > 0$ and let s be a non-negative number. The *diameter* of a non empty set U of Ω is defined as $|U| = \sup\{d(x, y) : x, y \in U\}$ and if a subset A of Ω is such that $A \subseteq \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, the class $\{U_i\}$ is called a δ-cover of A.

The *Hausdorff s-dimensional outer measure* of A, denoted by $h^s(A)$, is defined on $\wp(\Omega)$, the class of all subsets of Ω, as

$$h^s(A) = \lim_{\delta \to 0} \inf \sum_{i=1}^{+\infty} |U_i|^s .$$

where the infimum is over all δ-covers $\{U_i\}$.

A subset A of Ω is called *measurable* with respect to the outer measure h^s if it decomposes every subset of Ω additively, that is if $h^s(E) = h^s(A \cap E) + h^s(E - A)$ for all sets $E \subseteq \Omega$.

Hausdorff s-dimensional outer measures are submodular, continuous from below and their restriction on the Borel σ-field is countably additive.

The *Hausdorff dimension* of a set A, $dim_H(A)$, is defined as the unique value, such that

$$h^s(A) = +\infty \text{ if } 0 \leq s < dim_H(A),$$
$$h^s(A) = 0 \text{ if } dim_H(A) < s < +\infty.$$

For every $B \in$ **B** denote by s the Hausdorff dimension of B and let h^s be the Hausdorff s-dimensional Hausdorff outer measure associated to the coherent upper conditional prevision. For every bounded random variable X a coherent upper conditional prevision $\overline{P}(X|B)$ is defined by the Choquet integral with respect to its associated Hausdorff outer measure if the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension. Otherwise if the conditioning event has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity it is defined by a 0-1 valued finitely, but not countably, additive probability.

Theorem 1 *Let (Ω, d) be a metric space and let **B** be a partition of Ω. For every $B \in$ **B** denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m_B be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Thus, for each $B \in$ **B**, the function defined on $\wp(B)$ by*

$$\overline{P}(A|B) = \begin{cases} \frac{h^s(A \cap B)}{h^s(B)} & if \quad 0 < h^s(B) < +\infty \\ m_B & if \quad h^s(B) \in \{0, +\infty\} \end{cases}$$

is a coherent upper conditional probability.

If $B \in$ **B** is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s the fuzzy measure μ_B^* defined for every $A \in \wp(B)$ by $\mu_B^*(A) = \frac{h^s(AB)}{h^s(B)}$ is a coherent upper conditional probability, which is submodular, continuous from below and such that its restriction to the σ-field of all μ_B^* measurable sets is a Borel regular countably additive probability.

The coherent upper unconditional probability $\overline{P} = \mu_\Omega^*$ defined on $\wp(\Omega)$ is obtained for B equal to Ω.

Denoted by h_s the Hausdorff inner measure of order s, which is the dual of the Hausdorff outer measures of order s h^s, we have that the conjugate lower conditional probability $\overline{\mu}_B^*$ of μ_B^* is

$$\overline{\mu}_B^*(B) = \mu_B^*(\Omega) - \mu_B^*(B^c) = 1 - 0 = 1 = \mu_B^*(B)$$

and since

$$\mu_B^*(\Omega) - \mu_B^*(B^c) = \frac{h^s(\Omega \cap B)}{h^s(B)} - \frac{h^s(B^c \cap B)}{h^s(B)} = \frac{h_s(B)}{h^s(B)}$$

so that $\frac{h_s(B)}{h^s(B)} = 1$ and every B is μ_B^*-measurable, i.e. $h_s(B) = h^s(B)$.

Moreover

$$\overline{\mu}_B^*(A) = \mu_B^*(\Omega) - \mu_B^*(A^c) = \frac{h^s(\Omega \cap B)}{h^s(B)} - \frac{h^s(A^c \cap B)}{h^s(B)} = \frac{h_s(A \cap B)}{h_s(B)}.$$

In the following theorem the coherent upper conditional probability defined in Theorem 1 is extended to the class of all bounded random variables and when the conditioning event B has positive and finite Hausdorff outer measure in its Hausdorff dimension the coherent upper prevision id defined by the Choquet integral (Choquet 1953).

Theorem 2 *Let (Ω, d) be a metric space and let **B** be a partition of Ω. For every $B \in$ **B** denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m_B be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Then for each $B \in$ **B** the functional $\overline{P}(X|B)$ defined on $L(B)$ by*

$$\overline{P}(X|B) = \begin{cases} \frac{1}{h^s(B)} \int_B X dh^s & if \quad 0 < h^s(B) < +\infty \\ m_B & if \quad h^s(B) \in \{0, +\infty\} \end{cases}$$

is a coherent upper conditional prevision.

When the conditioning event B has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity, an additive conditional probability is coherent if and only if it takes only $0 - 1$ values. Because linear previsions on $L(B)$ are uniquely determined by their restrictions to events, the class of linear previsions on $L(B)$ whose restrictions to events take only the values 0 and 1 can be identified with the class of $0 - 1$ valued additive probability defined on all subsets of B (Walley 1991). In Theorem 1 and Theorem 2 a different m_B is chosen for each B.

If the conditioning event B has positive and finite Hausdorff outer measure in its Hausdorff dimension the functional $\overline{P}(X|B)$ is proven to be monotone, comonotonically additive, submodular and continuous from below.

Preference ordering and indifference between random variables represented by coherent lower and upper conditional previsions

Non-linear functional are used to represent preference orderings that cannot be represented by a linear functional.

Definition 2 *A preference ordering $\succ$ on the class $L(B)$ of random variables defined on B is represented by a linear functional Γ if and and only if*

$$X_i|B \succ X_j|B \Leftrightarrow \Gamma(X_i|B) > \Gamma(X_j|B)$$
and
$$X_i|B \approx X_j|B \Leftrightarrow \Gamma(X_i|B) = \Gamma(X_j|B)$$

Example 1 *Let $\Omega = N$ and let $\boldsymbol{B} = \{B_1, B_2\}$ be the partition of Ω where $B_1 = \{p \in N : p = 2n; n \in N\}$ and $B_2 = \{d \in N : d = 2n - 1; n \in N\}$. Let μ be a probability measure defined on the field generated by $\boldsymbol{B}$. Let consider the class $K = \{X_1, X_2, X_3\}$ of bounded $\boldsymbol{B}$-measurable random variables defined on Ω by*

random variables	B_1	B_2
X_1	*0.3*	*0.3*
X_2	*0.7*	*0.0*
X_3	*0.0*	*0.7*

The preference ordering $X_1 \succ X_2$ and $X_2 \approx X_3$ cannot be represented by the linear functional (weighted sum) $\Gamma(X) = \sum_{i=1}^{2} X(B_i)\mu(B_i)$ since there exists no probability measure μ such that the following system has solution:

$$\begin{cases} X_1 \succ X_2 \\ X_2 \approx X_3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0.3\mu(B_1) + 0.3\mu(B_2) > 0.7\mu(B_1) + 0.0\mu(B_2) \\ 0.7\mu(B_1) + 0.0\mu(B_2) = 0.0\mu(B_1) + 0.7\mu(B_2). \end{cases}$$

Let $\overline{P}(X|B)$ a coherent upper conditional prevision and let $\underline{P}(X|B)$ its conjugate lower conditional prevision.

A partial strict order, which is an antisymmetric and transitive binary relation between random variables, can be represented by the lower conditional prevision $\underline{P}(X|B)$.

Definition 3 *We say that X is preferable to Y given B with respect to $\underline{P}$, i.e. $X \succ_* Y$ in B if and only if*

$$\underline{P}((X - Y)|B) > 0$$

In particular we show that the binary relation $\succ_*$ satisfies the antisymmetric property, i.e.

$$X \succ_* Y \Longleftrightarrow \underline{P}((X - Y)|B) > 0 \Longrightarrow$$
$$\underline{P}((Y - X|B) \leq 0 \Longleftrightarrow Y not \succ_* X.$$

In fact

$$0 < \underline{P}((X - Y)|B) < \overline{P}((X - Y)|B) \Longrightarrow$$
$$\overline{P}((X - Y)|B) = -\underline{P}((Y - X)|B) > 0$$

so that $\overline{P}((Y - X)|B) < 0$ that is $Y not \succ_* X$.

Two random variables which have previsions equal to zero cannot be compared by the ordering $\succ_*$.

A binary relation $\propto$ can be defined on $L(B)$ with respect to $\overline{P}$ but it cannot represent a strict preference ordering because it does not satisfied the antisymmetric property.

Definition 4 *We say that $X \propto Y$ given B if and only if $\overline{P}((X - Y)|B) > 0$.*

Example 2 *Let $X, Y \in L(B)$ such that $\overline{P}((X-Y)|B) > 0$ and $\underline{P}((X - Y)|B) < 0$; then*

$\overline{P}((X - Y)|B) > 0$ *does not imply* $\overline{P}((Y - X)|B) < 0$ *since*

$\overline{P}((Y - X)|B) < 0 \Longleftrightarrow -\underline{P}((X - Y)|B) < 0 \Longleftrightarrow \underline{P}((X - Y)|B) > 0$

Two complete equivalence relations, which are complete reflexive, symmetric and transitive binary relations on $L(B)$ can be represented by the coherent upper conditional prevision $\overline{P}(X|B)$.

Definition 5 *Two random variables X and $Y \in L(B)$ are equivalent given B with respect to $\overline{P}$ if and only if $\overline{P}(X|B) = \overline{P}(Y|B)$.*

Definition 6 *We say that X and Y are indifferent given B with respect to $\overline{P}$, i.e. $X \approx Y$ in B if and only if*

$$\overline{P}((X - Y)|B) = \overline{P}((Y - X)|B) = 0.$$

Remark 1 *If the coherent conditional prevision $P(\cdot|B)$ is linear then $P((X - Y)|B) = P((Y - X)|B) = 0 \Longleftrightarrow P(X|B) = P(Y|B)$*

and two random variables X and Y are indifferent given B if and only if they are equivalent given B.

Theorem 3 *Let $X, Y \in L(B)$ be two random variables, which are indifferent given B with respect to $\overline{P}$ then they are indifferent with respect to the conjugate lower conditional prevision $\underline{P}$, that is $\underline{P}((X - Y)|B) = \underline{P}((Y - X)|B) = 0$.*

Proof. Since $X, Y \in L(B)$ are indifferent given B we have

$$-\underline{P}((Y - X)|B) = \overline{P}((X - Y)|B) = 0$$
$$-\underline{P}((X - Y)|B) = \overline{P}((Y - X)|B) = 0,$$

so that $\underline{P}((X - Y)|B) = \underline{P}((Y - X)|B) = 0.\diamond$

Theorem 4 *Let $X, Y \in L(B)$ be two random variables, such that $X \succ_* Y$ given B with respect to $\underline{P}(\cdot|B)$ then X and Y are not indifferent given B with respect to $\overline{P}(\cdot|B)$.*

Proof. If $X \succ_* Y$ given B with respect to $\underline{P}(\cdot|B)$ then

$0 < \underline{P}((X-Y)|B) \leq \overline{P}((X-Y)|B)$ so $\overline{P}((X-Y)|B) \neq 0$ and X and to Y are not indifferent given B with respect to $\overline{P}(\cdot|B)$ according to Definition 4. ⋄

In the next example it is shown that the lower vacuous conditional prevision does not represent the preference ordering $X_1 \succ_* X_2$ and the upper vacuous conditional prevision does not represent the indifference between X_2 and X_3

Example 3 *Let **B** and $K = \{X_1, X_2, X_3\}$ as in Example 1. The preference ordering $X_1 \succ X_2$ and $X_2 \approx X_3$ cannot be represented by the lower vacuous conditional prevision defined by $\underline{P}(X|\Omega) = inf\,\{X(\omega) : \omega \in \Omega\}$ since*

$$\underline{P}((X_1 - X_2)|\Omega) = -0.4 \textit{ and}$$
$$\underline{P}((X_2 - X_3)|\Omega) = \underline{P}(X_3 - X_4|\Omega) = 0$$

and is is not represented by the upper vacuous conditional prevision $\overline{P}(X|\Omega) = sup\,\{X(\omega) : \omega \in \Omega\}$ because

$$\overline{P}((X_1 - X_2)|\Omega) = 0 \textit{ and}$$
$$\overline{P}((X_2 - X_3)|\Omega) = \overline{P}((X_3 - X_2)|\Omega) = 0.7.$$

Let $\overline{P}(X|B)$ the coherent upper conditional prevision defined in Theorem 2 and let $\underline{P}(X|B)$ its conjugate lower conditional prevision.

Example 4 *Let (Ω, d) and be **B** as in Example 1 and let (Ω, d) be a metric space. So $dim_H(\Omega) = 0$ and $h^0 = (\Omega) = +\infty$, $dim_H(B_1) = dim_H(B_2) = 0$ and $h^0(B_1) = h^0(B_2) = +\infty$.*

By Theorem 2 we have the coherent lower and upper conditional prevision are equal to a 0-1 valued finitely additive, but not countably, probability

$P((X_1 - X_2)|\Omega) = 1$

$P((X_2 - X_3|\Omega)) = P(X_2 - X_3|\Omega) = 0.$

and the ordering $X_1 \succ X_1$ and $X_2 \approx X_2$ can be represented by the given coherent conditional prevision.

The model and the brain's activity

In the section the given model is investigated to describe the human decision-making that is influenced by conscious and unconscious aspects.

For Matte Blanco (Matte Blanco 1975), the conscious and unconscious are two different modes of being, asymmetric and in becoming the first, symmetric and static the second. The author produced a description of the structure and functioning of the unconscious with the purpose to account for the non-logical aspects of human thought. He drew a distinction between the logical conscious thought, structured on the categories of time and space and ruled by the Aristotle's principle of non-contradiction, which he defined "asymmetrical thought", and the unconscious thought, which he defined "symmetrical thought", based upon the principle of symmetry and the principle of generalization. According to the author, both types of thoughts combine in the different experiences of human thinking thus yielding to a bi-logic asset. Emotions are the way to reach the unconscious, they function the same way as the unconscious and are the means to decode it.

Coherent lower conditional prevision could be used to represent the partial strict preference order which is the result of the conscious thought and coherent upper conditional prevision could represent the equivalence assigned by the unconscious thought. According to this interpretation by Theorem 3 we could conclude that if two random variables are indifferent with respect to the unconscious thought then one of them cannot be preferable to the other with respect to the conscious thought; by Theorem 4 we could obtain that if a random variable is preferable to another one with respect to the conscious mind then they cannot be indifferent with respect to the unconscious thought.

Pathological situations can be obtained when two random variables are indifferent with respect to the unconscious thought but one of them if preferable to the other one with respect to the conscious thought. They can be captured by the model because in this case the lower conditional prevision should be greater than the upper prevision.

The updating model based on Hausdorff outer and inner measure can represent respectively the awareness process of the unconscious and conscious thought which depend on unexpected events in all cases.

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